The process of differentiation may be carried out any number of times with respect to different parameters using formulae of differentiation analogous to the above. When the various derivatives are added together the result indicates that the natural generalisation of a series of spherical harmonics of form

$$\frac{S_0(\theta,\phi)}{r} + \frac{S_1(\theta,\phi)}{r^2} + \frac{S_2(\theta,\phi)}{r^3} +$$

is the following type of series of Hertzian functions of different orders

$$\mathbf{G} = \frac{\mathbf{g}_0}{\nu} + \operatorname{div}\left(\frac{\mathbf{a}_0\mathbf{a}_1}{\nu}\right) + \operatorname{div}\operatorname{div}\left(\frac{\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2}{\nu}\right) + \operatorname{div}\operatorname{div}\operatorname{div}\left(\frac{\mathbf{c}_0\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3}{\nu}\right) + .$$

Here $\mathbf{g}_0$, $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$ .... are arbitrary vector functions of α. It should be remarked that the vector with suffix n is treated as the vector in forming the n divergence while the other vectors are treated for the moment as scalar quantities. The product of k vectors which occurs in the $(k + 1)$th term is to be regarded as a tensor of the kth order with k components each of which is a product of components of the separate vectors; there appear to be enough arbitrary functions in a sum of products of this type with $k = 0, 1, 2, \ldots, K$ for the representation of the sum of a number of Hertzian functions up to order K.

[1] As each shell of electricity moves outwards it induces a secondary separation of electricity so that electricity flows back to a new position of the primary singularity (ξ, η, ζ) and tends to maintain the electric separation. The volume density of the compensating electricity created at the primary singularity isthus not ρ but is proportional to Ψ/r, it is this electricity which is regarded as forming the elementary æther associated with the primary singularity and it is this electricity which, on account of its displacement from the concentrated charge, is directly responsible for the field.

[2] See for instance Wilson, E. B., *Washington Acad. Sci.*, **6**, 1916, (665–669).

[3] Larmor, J., *London, Proc. Mathe. Soc.*, **13**, 1913, p. 51.

[4] Whitehead, A. N., *The Anatomy of some Scientific Ideas, The Organization of Thought*, London, 1917, p. 182.

INVARIANTS WHICH ARE FUNCTIONS OF PARAMETERS OF THE TRANSFORMATION

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A systematic theory and interpretation of invariantive functions which contain the parameters of the linear transformations to which a quantic

f_m of order m is subjected has not been formulated, although a paper on invariants[1] published in 1843 by Boole treated certain functions of this type. These were the concomitants of forms under transformations which rotate cartesian axes inclined at an angle w into another set with inclination w'; invariants which contain the parameter w. Orthogonal concomitants, which are special cases with $w = \frac{1}{2}\pi$, were made the subject of a number of later papers[2], notably by Elliott and by MacMahon.

I have considered a general doctrine of such concomitants for the transformation with four parameters

$$T\colon\ x_1 = \alpha_1 x_1' + \alpha_2 x_2',\ x_2 = \beta_0 x_1' + \beta_1 x_2';\quad D = \alpha_1\beta_1 - \alpha_2\beta_0 \neq 0.$$

The elements of the methods are based upon the two forms

$$\xi = 2\beta_0 x_1 + (\beta_1 - \alpha_1 + \Delta)x_2,\quad \eta = 2\beta_0 x_1 - (\beta_1 - \alpha_1 - \Delta)x_2;$$

whose roots are the poles of T, and the expansion of f_m in terms of ξ, η as arguments. The quantity Δ employed here is the square root of the discriminant of the form

$$J\colon \beta_0 x_1^2 + (\beta_1 - \alpha_1)x_1x_2 - \alpha_2 x_2^2.$$

The coefficients φ_{m-2i} $(i = 0, \ldots, m)$ in the expansion of f_m are invariants of a new type[3] belonging to the domain $R(1, T, \Delta)$ of rational polynomials in the coefficients of f_m and those of T, increased by adjunction of Δ. These invariants compose a fundamental system in R. They satisfy the invariant relations

$$\varphi'_{m-2i} = \rho^{m-2i} D^i \varphi_{m-2i}\ (i = 0, \ldots, m), \tag{1}$$

in which ρ is one of the two factors of D in R:

$$\rho = \tfrac{1}{2}(\alpha_1 + \beta_1 + \Delta). \tag{2}$$

When one seeks complete systems for the given domain $R(1, T, 0)$, free from Δ but including rationally the coefficients of T, it is found that the concomitants are in one to one correspondence with those invariantive products

$$P = \prod_{i=0}^{m} \varphi_{m-2i}^{x_i}\, \xi^{\alpha}\, \eta^{\beta}$$

for which the exponent of ρ in the invariant relation $P' = \rho^a D^b P$, is zero. The conclusion is then drawn that concomitants in $R(1, T, 0)$ are in one to one correspondence with the solutions of the diophantine equation

$$\delta \equiv \sum_{i=0}^{m} x_i\,(m - 2i) + \beta - \alpha = 0. \tag{3}$$

This infinitude of concomitants therefore forms a system which possesses the property of finiteness,[4] and the fundamental invariants are furnished by the finite set of irreducible solutions of $\delta = 0$.

In the ternary realm the lines joining the three poles of the transformation T on three variables furnish three linear forms in terms of which any quantic f_m in three variables can be expanded. The coefficients in this expansion urnish complete systems in each of several domains. In particular, if T is the transformation which rotates cartesian axes in three dimensional space,

$$\begin{aligned} x &= l_1x' + l_2y' + l_3z', \\ y &= m_1x' + m_2y' + m_3z', \\ z &= n_1x' + n_2y' + n_3z', \end{aligned}$$

the coefficients being the well-known direction cosines of three axes, the invariant triangle on the poles consists of the lines

$$\begin{aligned} f_{\pm 1} &\equiv (l_3 + n_1 e^{\pm i\theta})\, x + (m_3 + n_2 e^{\pm i\theta})\, y + (n_3 - \overline{l_1 + m_2} e^{\pm i\theta} + e^{\pm 2i\theta})\, z = 0, \\ f_0 &\equiv (l_3 + n_1)\, x + (m_3 + n_2)\, y + (n_3 - l_1 - m_2 + 1)\, z = 0, \end{aligned}$$

where θ is a definite auxiliary angle. The coefficients in the expansion of f_m in the arguments $f_{\pm 1}$, f_0 are invariants belonging to the domain of complex numbers, while the finiteness of complete systems in the real domain is determined, and the fundamental concomitants are given, by the finite set of irreducible solutions of the linear diophantine equation

$$\sum_{i=0}^{m} \sum_{j=0}^{m-i} x_j^{(i)} (m - i - 2j) + \beta - \alpha = 0. \tag{4}$$

The Invariants of Relativity.—Among numerous important particular cases of the above theory is the transformation of space and time coordinates in the theory of relativity, known as the transformation of Einstein.[5] This consists of

$$T_1: \ t = \mu (c^2t' + vx') / c, \ \ x = \mu (vt' + x') c, \ \ y = y', \ \ z = z',$$

where $\mu = 1/(c^2 - v^2)^{-\frac{1}{2}}$, t is the time, c the velocity of light, v the relative velocity of the moving systems of reference and x, y, z space coördinates. For these unitary substitutions, $\xi = ct + x$, $\eta = ct - x$ and the invariant relations are

$$\xi' = \rho\xi, \ \eta' = \rho^{-1}\eta,$$

in which (Cf. (2))

$$\rho = \sqrt{c - v} / \sqrt{c + v}.$$

We now find

$$J : c^2t^2 - x^2,$$

this being an absolute universal covariant of T_1 for all values of the relative velocity V.

A binary form $f_{1m}(t, x)$ in t and x, whose coefficients are constants or arbitrary functions of the quantities left fixed by T_1, has a finite system of non-absolute invariants corresponding to the forms in the system for f_m belonging to R $(1, T, \Delta)$, and a finite system of absolute concomitants analogous to the system for f_m in the domain R $(1, T, 0)$. To obtain the concomitants of $f_{1m}(t, x)$ one may either particularize those of f_m under T, making the substitutions which reduce T to T_1, or, the invariants under Einstein's transformations can be developed *ab initio* by the methods described above for T, the arguments of the expansion of f_{1m} being now $\xi = ct + x$, $\eta = ct - x$. These invariantive functions represent invariant loci in four dimensional space if the time t is interpreted as a fourth dimension. All are free from v.

A paper in which the above theory and applications are developed in detail and which includes tables of the relativity invariants computed for the general f_{1m} in the case of the non-absolute systems, and for the orders 1 to 3, inclusive, in the case of the absolute systems, is to appear in the *Annals of Mathematics*.

[1] Boole, *Cambridge Mathematical Journal*, **3**, 1843, (1).

[2] Elliott, *London*, *Proc. Math. Soc.*, **33**, 1901, (226).

[3] O. E. Glenn, *New York*, *Trans. Amer. Math. Soc.*, **18**, 1917, (443)

[4] Hilbert, *Leipzig*, *Math. Ann.*, **36**, 1890, (473).

[5] Einstein, *Leipzig*, *Ann. Physik*, **17**, 1905. Lorentz, Einstein, and Minkowski, *Das Relativitätsprinzip*, 1913, p. 27. R. D. Carmichael, *The Theory of Relativity*, 1913, p. 44.